MATH 2050C Lecture on 3/20/2020

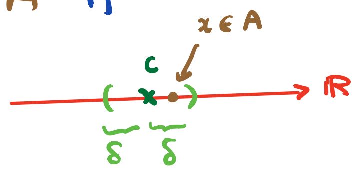
[Announcement: PS 7 due today, PS 8 posted.]

GOAL: Define $\lim_{x \to c} f(x)$ for a function $f: A \subseteq \mathbb{R} \to \mathbb{R}$

where c is a cluster pt. of A [$f(x) \approx L$ as $x \approx c$ & $x \in A$]

Def$^{\underline{n}}$: Given $A \subseteq \mathbb{R}$, we say $c \in \mathbb{R}$ is a cluster pt. of A if

$\forall\, \delta > 0$, $\exists\, x \in A$ s.t. [$x \neq c$ and $|x - c| < \delta$]



$$\Big[\quad 0 < |x - c| < \delta \quad \Big]$$

Remark: $c \in A$ OR $c \notin A$ (either is possible)

Prop: Let $A \subseteq \mathbb{R}$ be given.

$c \in \mathbb{R}$ is a cluster pt. of A $\iff$ $\exists$ seq. (a_n) in A s.t. $\begin{cases} a_n \neq c & \forall\, n \in \mathbb{N} \\ \lim (a_n) = c \end{cases}$

Proof: "$\Rightarrow$" Assume $c \in \mathbb{R}$ is a cluster pt. of A.

Take for each $n \in \mathbb{N}$, $\delta_n := \frac{1}{n} > 0$.

By def$^{\underline{n}}$, $\exists\, a_n \in A$ st. $a_n \neq c$ and $|a_n - c| < \delta_n = \frac{1}{n}$.

By squeeze thm, $\lim (a_n) = c$.

"$\Leftarrow$" Exercise.

$\blacksquare$

★ Def$^{\underline{n}}$ ★: Let $f: A \subseteq \mathbb{R} \to \mathbb{R}$ be a function.

Suppose $c \in A$ is a cluster point of A.

We say that f converges to $L \in \mathbb{R}$ at c, written "$\lim_{x \to c} f(x) = L$"

iff $\forall\, \varepsilon > 0$, $\exists\, \delta = \delta(\varepsilon) > 0$ s.t.

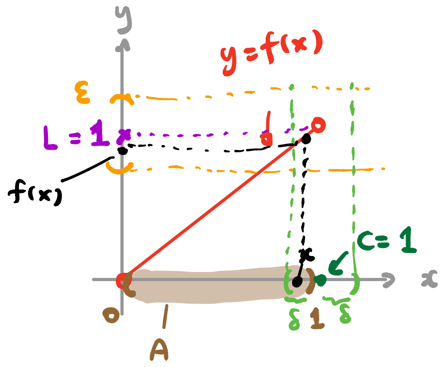
$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad x \in A \text{ and } 0 < |x - c| < \delta$$

(ie. $x \neq c$)

Remarks: (1) $\lim_{n\to\infty}(x_n) = L$ BUT $\lim_{x\to c} f(x) = L$ for any c cluster pt.

(2) f needs NOT be defined at $x = c$.

E.g.) Consider $f : A = (0,1) \to \mathbb{R}$, $f(x) := x \quad \forall x \in (0,1)$.



$$\lim_{x\to 1} f(x) = 1$$

Note: $f(1)$ not defined since $1 \notin A$

Let's evaluate some "simple" limits using the definition.

Example 1: $\boxed{\lim_{x\to c} x = c}$

Here, $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) := x \quad \forall x \in \mathbb{R}$.

$\lim_{x\to c} f(x)$

[Caution: This is a different function from above as domains are different.]

Proof: Let $\varepsilon > 0$. Choose $\delta = \varepsilon/2 > 0$.

Then, $\forall x \in A = \mathbb{R}$, $0 < |x - c| < \delta$, we have

$$|f(x) - c| = |x - c| < \delta < \varepsilon$$

Want: ✓

Example 2: $\boxed{\lim_{x\to c} x^2 = c^2}$

$f(x) := x^2$

$f : \mathbb{R} \to \mathbb{R}$

Proof: Let $\varepsilon > 0$. Choose $\delta = \min\left\{1, \frac{\varepsilon}{1+2|c|}\right\} > 0$.

Then, $\forall x \in \mathbb{R}$ s.t. $0 < |x - c| < \delta$, we have

First of all,

$$|x - c| \le 1 \implies |x| \le |x - c| + |c| \le 1 + |c|$$

Thus,

$$\left| x^2 - c^2 \right| = |x+c| \cdot \underbrace{|x-c|}_{\text{small}} \leq (|x|+|c|)\cdot|x-c| \leq (1+2|c|)\cdot|x-c|$$

hope: bdd.

Squeeze out a term involving $|x-c|$

$$< (1+2|c|)\cdot\delta \leq \varepsilon$$

$\square$

Example 3: $\boxed{\lim_{x\to c} \frac{1}{x} = \frac{1}{c}}$

- $f(x) := \frac{1}{x}$ $\quad f: A = \{x \in \mathbb{R} \mid x \neq 0\} \to \mathbb{R}$
- provided that $c \neq 0$.

Proof: Exercise (c.f. above example & $\lim_{n\to\infty}\left(\frac{1}{y_n}\right) = \frac{1}{\lim_{n\to\infty}(y_n)}$).

Example 4: $\boxed{\lim_{x\to 2} \frac{x^3-4}{x+1} = \frac{4}{3}}$

- $f(x) := \frac{x^3-4}{x+1}$ $\quad f: \mathbb{R}\setminus\{-1\} \to \mathbb{R}$

$|x-2|<\delta$

Proof: Let $\varepsilon > 0$.

Choose $\delta = \min\left\{1, \frac{3\cdot 2}{10000}\varepsilon\right\} > 0$.

Note: $|x-2| \leq 1 \Rightarrow 1 \leq x \leq 3$

$\Rightarrow 2 \leq x+1 \leq 4$

$\Rightarrow |x+1| \geq 2$.

Note: $|x-2| \leq 1 \Rightarrow |x| \leq 3$

$\Rightarrow |3x^2+6x+8| \leq 3\cdot 3^2 + 6\cdot 3 + 8 < 10000$

For any $x \in \mathbb{R}$ s.t. $0 < |x-2| < \delta$,

we have

$$\left| \frac{x^3-4}{x+1} - \frac{4}{3} \right| = \boxed{\frac{|3x^2+6x+8|}{3|x+1|}} \cdot |x-2|$$

$$< \frac{10000}{3\cdot 2}\delta \leq \varepsilon$$

$\square$

$$\left| \frac{x^3-4}{x+1} - \frac{4}{3} \right|$$

$$= \left| \frac{3(x^3-4) - 4(x+1)}{3(x+1)} \right|$$

$$= \left| \frac{3x^3 - 4x - 16}{3(x+1)} \right|$$

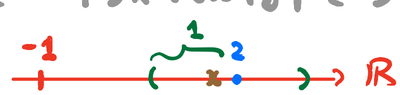
$$= \frac{|(3x^2+6x+8)\cdot(x-2)|}{|3(x+1)|}$$

bdd?

$$= \underbrace{\frac{|3x^2+6x+8|}{|3(x+1)|}}_{\text{bdd}} \cdot \underbrace{|x-2|}_{\text{Small}}$$

bdd from 0

- $|3x^2+6x+8| \leq 3|x|^2 + 6|x| + 8$ O.K. if $|x| \leq ?$ when $x \approx 2$



- $|x+1| \geq 1$ when $x \approx 2$

Lemma: Limit of a function, if exists, is unique.

[**Ex:** Prove this.]

[**Basic Philosophy:** Many facts about limits of seq. have an analogue for limits of functions.]

Q: Why?

A: These two kinds of limits are related by the next thm:

Thm: (Sequential Criterion) $f: A \to \mathbb{R}$. c is a cluster pt of A

$$\lim_{x\to c} f(x) = L \iff \forall \text{ seq. } (x_n) \text{ in } A \text{ s.t. } \begin{cases} x_n \neq c & \forall n \in \mathbb{N} \\ \lim(x_n) = c \end{cases}$$

we have $\lim_{n\to\infty}(f(x_n)) = L$

limit as function

limit of sequences

Proof: "$\Rightarrow$" Assume $\lim_{x\to c} f(x) = L$.

Let (x_n) be a seq. in A s.t. $x_n \neq c$ $\forall n$ & $\lim(x_n) = c$.

Claim: $\lim_{n\to\infty}(f(x_n)) = L$.

Pf: Let $\varepsilon > 0$.

Since $\lim_{x\to c} f(x) = L$, by def^{n}, $\exists\, \delta = \delta(\varepsilon) > 0$ s.t.

$$|f(x) - L| < \varepsilon, \quad \forall\, 0 < |x - c| < \delta$$

Since $\lim(x_n) = c$, by def^{n}, $\exists\, K = K(\delta) \in \mathbb{N}$ s.t.

$$0 < |x_n - c| < \delta \quad \forall\, n \geq K.$$

So, $|f(x_n) - L| < \varepsilon \quad \forall\, n \geq K$ $\square$

"$\Leftarrow$" By contradiction. Suppose $\lim_{x\to c} f(x) \neq L$.

free

$\Rightarrow \exists\, \varepsilon_0 > 0$, $\forall\, \delta > 0$, $\exists\, x_\delta \in A$ s.t.

$0 < |x_\delta - c| < \delta$ BUT $|f(x_\delta) - L| \geq \varepsilon_0$

Take $\delta_n := \frac{1}{n}$, $n \in \mathbb{N}$. Get $x_n \in A$ s.t.

$\underbrace{0 < |x_n - c| < \frac{1}{n}}$ <u>BUT</u> $\underbrace{|f(x_n) - L| \geq \varepsilon_0}$. $\forall n \in \mathbb{N}$

$\Rightarrow$ $\lim_{n \to \infty} (x_n) = c$ <u>BUT</u> $\lim_{n \to \infty} (f(x_n)) \neq L$

and $x_n \neq c \ \forall n$

<u>Contradiction!</u>